

Problem Statement

Suppose that you have ten light bulbs, that the lifetime of each is independent of all the other lifetimes, and that each lifetime has an exponential distribution with parameter λ .

- a. What is the probability that all ten bulbs fail before time t ?
- b. What is the probability that exactly k of the ten bulbs fail before time t ?
- c. Suppose nine of the bulbs have lifetimes that are exponentially distributed with parameter λ and the remaining bulb has a lifetime that is exponentially distributed with parameter μ (it is made by another manufacturer). What is the probability that exactly five of the ten bulbs fail before time t ?

Solution

- a. What is the probability that all ten bulbs fail before time t ?

This problem has 10 random variables.

X_1 =time of life of light bulb 1 with parameter λ_1
 X_2 =time of life of light bulb 2 :
 . .
 : .
 X_{10} =time of life of light bulb 10 with parameter λ_{10}

Where each random variable is distributed exponentially

$$\begin{aligned} X_1 &\sim \text{Exp}(\lambda_1) \\ X_2 &\sim \text{Exp}(\lambda_2) \\ &\vdots \\ X_{10} &\sim \text{Exp}(\lambda_{10}) \end{aligned}$$

We know the CDF of the exponential to be:

$$F_x(x_n) = 1 - \exp(-\lambda_n x) \quad \text{where } n=1,2,\dots,10$$

The question asks what is $P(X_1, \dots, X_{10} < t)$?
 Since each random variable is independent of each other

$$P(X_1 < t, X_2 < t, \dots, X_{10} < t) = P(X_1 < t) * P(X_2 < t) * \dots * P(X_{10} < t)$$

Knowing $P(X_1 < t) = F_x(t) = 1 - \exp(-\lambda_1 * t)$

$$P(X_1 < t, X_2 < t, \dots, X_{10} < t) = [1 - \exp(-\lambda_1 * t)] * [1 - \exp(-\lambda_2 * t)] * \dots * [1 - \exp(-\lambda_{10} * t)]$$

Assuming each I is identically distributed ($I_1 = I_2 = \dots = I_{10}$)

$$P(X_1 < t, X_2 < t, \dots, X_{10} < t) = [1 - \exp(-I * t)]^{10}$$

b. What is the probability that exactly k of the ten bulbs fail before time t ?

For this problem will we define a new random variable

Y=number of bulbs that fail before time t out of 10 bulbs

Where the probability of a single bulb failing is equal to the probability found in part **a** for only one bulb, instead of 10.

$$Y \sim \text{bin}(n=10, p=1 - \exp(-I * t))$$

Where the pdf of Y is defined as

$$f_Y(y) = \binom{10}{y} (1 - \exp(-I t))^y \exp(-I t)^{10-y}$$

The question asks what is $P(Y=k)$?

$$P(Y=k) = \binom{10}{y} (1 - \exp(-I * t))^k (1 - (1 - \exp(-I * t)))^{10-k}$$

$$P(Y=k) = \binom{10}{y} (1 - \exp(-I * t))^k (\exp(-I * t))^{10-k}$$

c. Suppose nine of the bulbs have lifetimes that are exponentially distributed with parameter I and the remaining bulb has a lifetime that is exponentially distributed with parameter q (it is made by another manufacturer). What is the probability that exactly five of the ten bulbs fail before time t ?

Y=number of bulbs that fail in time t , out of 10, where 9 bulbs have lifetimes that are exponentially distributed with parameter I and 1 bulb is exponentially distributed with parameter q .

As with in problem **b**, Y is distributed binomially. The probability that a single bulb fails in time t is $1 - \exp(-I * t)$ or $1 - \exp(-q * t)$, depending on the parameter. So the probability that a light bulb does not fail in time t is $\exp(-I * t)$ or $\exp(-q * t)$.

The question asks what is $P(Y=5)$?

There are two ways that this could happen, either with all 5 bulbs that fail having a parameter of λ or 4 bulbs that fail having a parameter λ and the one bulb with parameter μ fails also. So we need to find both of those probabilities and add them together.

$$P(Y=5) = P(5 \lambda \text{'s fail}) + P(4 \lambda \text{'s and 1 } \mu \text{ fail})$$

$$P(5 \lambda \text{'s fail}) = \binom{9}{5} (1 - \exp(-\lambda t))^5 \exp(-\lambda t)^4 \exp(-\mu * t)$$

$$P(4 \lambda \text{'s and 1 } \mu \text{ fail}) = \binom{9}{4} (1 - \exp(-\lambda t))^4 \exp(-\lambda t)^5 (1 - \exp(-\mu * t))$$

$$P(Y=5) = \binom{9}{5} (1 - \exp(-\lambda t))^5 * \exp(-\lambda t)^4 * (\exp(-\mu * t)) +$$

$$\binom{9}{4} (1 - \exp(-\lambda t))^4 * (1 - \exp(-\mu * t)) * (\exp(-\lambda t))^5$$