PHYX 450

Final Exam

Due Friday, May 13, 2005

Qualitative Questions

Answer the following questions without performing any complex calculations, but with as much qualitative physical detail as possible. Your objective is to demonstrate a mastery over the concepts presented in the course.

Question 1. You have most likely encountered ideas in this course that caused you great difficulty. Which idea or concept has been most difficult to understand and why it has caused you problems? Be as verbose as you'd like.

Question 2. Explain why the states of the hydrogen atom are so dependent on a choice of z–axis, even though the potential is spherically symmetric.

Question 3. How many independent eigenstates are there in the coupled representation for a 3–component system with $l_1 = 3$, $l_2 = 1$, $l_1 = 5$? Make a listing of all the possible values of the coupled angular momentum quantum number l.

Question 4. When are expectation values of physical quantities in a linear combination of stationary states independent of time?

Question 5. An electron and a proton of identical energy are incident on the same potential barrier. Is the probability of transmission for the electron greater than, less than, or equal to the transmission probability of the proton? Explain how you arrived at your answer.

Quantitative Questions

Question 6. Operator Fun!

a) Show that in the state $|lm\rangle$ that:

$$\left\langle L_x^2\right\rangle + \left\langle L_y^2\right\rangle = \hbar^2\left[l(l+1)-m^2\right] - \frac{1}{4}\left\langle\left\{\hat{L}_+, \hat{L}_-\right\}\right\rangle,$$

where $\left\{\hat{A}, \hat{B}\right\}$ is the anticommutator $\hat{A}\hat{B}+\hat{B}\hat{A}$.

b) Show that if $\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] = \left[\hat{B},\left[\hat{A},\hat{B}\right]\right] = 0$ then $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]}$. Outline of the proof:

i) Show that if we define a function $f(x) = e^{\hat{A}x}e^{\hat{B}x}$ then

$$\frac{df}{dx} = \left(\hat{A} + e^{\hat{A}x}\hat{B}e^{-\hat{A}x}\right)f(x)$$

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ii) Using the commutator relation $\left[\hat{B},\hat{A}^n\right] = n\hat{A}^{n-1}\left[\hat{B},\hat{A}\right]$, and Taylor expanding $e^{-\hat{A}x}$, show that $\left[\hat{B}, e^{-\hat{A}x}\right] = -e^{-\hat{A}x}\left[\hat{B},\hat{A}\right]x$.

iii) Combine results of parts (i) and (ii) to show that

$$\frac{df}{dx} = \left(\hat{A}+\hat{B}+\left[\hat{B},\hat{A}\right]x\right)f(x).$$

iv) Use the property $\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] = \left[\hat{B},\left[\hat{A},\hat{B}\right]\right] = 0$ to argue that $\hat{A}+\hat{B}$ and $\left[\hat{B},\hat{A}\right]$ may be treated as simple algebraic variables in the differential equation, allowing you to solve the equation as if they were numbers instead of operators.

v) Solve the ODE for $f(x)$ and set $x=1$.

Question 7. The Virial Theorem. An interesting result regarding potential and kinetic energies in central potentials of the form $V \propto 1/r$ is that $\langle V\rangle = -2\langle T\rangle$.

a) Derive the Virial Theorem by taking expectation values of $\hat{V} = \frac{Ze^2}{r}$ and $\hat{T} = \frac{\hat{p}_r^2}{2m} + \frac{\hat{L}^2}{2mr^2}$ in the arbitrary bound eigenstate $|nlm\rangle$. The orthonormality properties of the Laguerre polynomials given in Table 10.3 of Liboff may be helpful.

b) Argue that if the Virial Theorem holds in any bound eigenstate, it must hold in any arbitrary bound state.

Question 8. What do you do with a Tritium Atom? Tritium, ${}^3_1$H, is a radioactive isotope of hydrogen whose nucleus is composed of one proton and one neutron. *Note: Ignore all spin for this problem.*

a) Write down the functional form of the $|1S\rangle$ (ground) state of the electron orbiting the tritium nucleus, including numerical values for all constants in the expressions. What is the expectation of the energy, $\langle E \rangle$ in this state?

b) At time $t = 0$, the tritium nucleus spontaneously decays into a Helium–3 (${}^3_2$He) nucleus by the emission of an electron. Assuming that the emitted electron departs the system instantaneously and does not affect or interact with the original valence electron at all, argue that the state of the valence electron immediately after the decay is exactly as found in part (a).

c) Write down the functional form of the $|1S\rangle_{\text{He}}$ (ground), $|2S\rangle_{\text{He}}$ (first excited), and $|3S\rangle_{\text{He}}$ (second excited) S–states of an electron orbiting a ${}^3_2$He nucleus, including numerical values for all constants in the expressions.

d) Some time shortly after the nuclear decay, the total angular momentum of the valence electron is measured. What value(s) of L^2 may be measured, and with what probability?

e) The energy of the valence electron is then measured. What are the probabilities of finding the electron in the states $|1S\rangle_{\text{He}}$, $|2S\rangle_{\text{He}}$, and $|3S\rangle_{\text{He}}$?

Question 9. A Basic Spin Problem. A spin–1/2 particle is prepared in the state $|\psi\rangle = \frac{1}{\sqrt{5}}|\uparrow\rangle + i\frac{2}{\sqrt{5}}|\downarrow\rangle$, where the $|\uparrow\rangle$ and $|\downarrow\rangle$ kets are eigenstates of $\hat{S}_z$.

a) Suppose that the particle's spin is measured along the x–axis in this state. What is the probability that it is found to be spin–up?

b) Calculate $\langle S_x \rangle$ in this state.