

# Complete Arcs from Plane Curves

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## Finite Projective Planes and Arcs

The projective plane over  $\mathbb{F}_q$  may be realized as

$$\mathbb{P}^2(\mathbb{F}_q) = (\mathbb{F}_q^3 \setminus \{0\}) / \mathbb{F}_q^\times.$$

Its points are 1-dimensional subspaces of  $\mathbb{F}_q^3$ , and its lines are the zero loci of nonzero linear forms

$$aX + bY + cZ = 0.$$

- Each line contains  $q + 1$  points.
- Each point lies on  $q + 1$  lines.
- The total number of points is  $q^2 + q + 1$ .

A set of points  $A \subset \mathbb{P}^2(\mathbb{F}_q)$  is a  $d$ -arc if every line  $\ell$  satisfies

$$|A \cap \ell| \leq d.$$

A  $d$ -arc is *complete* if it is maximal among  $d$ -arcs under inclusion.

## 2. Curves as Sources of Arcs

Let  $C \subset \mathbb{P}^2$  be a smooth plane curve of degree  $d$  defined over  $\mathbb{F}_q$ .

For any line  $\ell \subset \mathbb{P}^2$ , Bézout gives

$$\deg(\ell \cap C) = d$$

counted with multiplicity over  $\overline{\mathbb{F}_q}$ .

Hence if  $\ell$  is not a component of  $C$ , then

$$|\ell(\mathbb{F}_q) \cap C(\mathbb{F}_q)| \leq d.$$

Therefore  $C(\mathbb{F}_q)$  is naturally a  $d$ -arc.

**Main question.** When is  $C(\mathbb{F}_q)$  already a *complete*  $d$ -arc?

### Example: Conics

Let  $C \subset \mathbb{P}^2$  be a smooth conic over  $\mathbb{F}_q$  so that  $C(\mathbb{F}_q)$  is a 2-arc

When  $q$  is odd,  $C(\mathbb{F}_q)$  is already complete. Let  $P \in \mathbb{P}^2(\mathbb{F}_q) \setminus C(\mathbb{F}_q)$ . We claim that some line through  $P$  meets  $C$  in two distinct  $\mathbb{F}_q$ -points.

Over  $\mathbb{F}_q$  with  $q$  odd, there are exactly two tangent lines from an external point to a smooth conic and no tangent lines from an internal point. In either case, among the  $q + 1$  lines through  $P$ , at least one is not tangent to  $C$ . Such a line intersects  $C$  in a degree-2 divisor defined over  $\mathbb{F}_q$ ; being nontangent, it consists of two distinct points, and because the line is defined over  $\mathbb{F}_q$ , those points are both  $\mathbb{F}_q$ -rational.

Thus every external point lies on a secant line to  $C(\mathbb{F}_q)$ , so  $C(\mathbb{F}_q)$  is a complete 2-arc.

### Problem Restated

For a smooth projective plane curve  $C$  over  $\mathbb{F}_q$  of degree  $d$ , the completeness of  $C(\mathbb{F}_q)$  means:

for every  $P \in \mathbb{P}^2(\mathbb{F}_q) \setminus C(\mathbb{F}_q)$ , some line through  $P$  meets  $C$  in  $d$  distinct  $\mathbb{F}_q$ -points.

## Projection from an External Point

Fix a smooth plane curve  $C/\mathbb{F}_q$  of degree  $d$  and an external point

$$P \in \mathbb{P}^2(\mathbb{F}_q) \setminus C.$$

Projection from  $P$  gives a morphism

$$\pi_P : C \rightarrow \mathbb{P}^1, \quad Q \mapsto \overline{PQ},$$

where  $\overline{PQ}$  is viewed as a point of the pencil of lines through  $P$ .

Since a general line through  $P$  meets  $C$  in  $d$  points,  $\pi_P$  is finite of degree  $d$ .

If  $X$  is an irreducible curve over  $\mathbb{F}_q$ , its function field  $\mathbb{F}_q(X)$  is the field of rational functions on  $X$ . A dominant morphism  $f : X \rightarrow Y$  induces an inclusion of function fields

$$f^* : \mathbb{F}_q(Y) \hookrightarrow \mathbb{F}_q(X), \quad \varphi \mapsto \varphi \circ f.$$

For finite morphisms of curves, this makes  $\mathbb{F}_q(X)$  into a finite field extension of  $\mathbb{F}_q(Y)$ , and

$$[\mathbb{F}_q(X) : \mathbb{F}_q(Y)] = \deg(f).$$

Applied to  $\pi_P$ , this gives a degree- $d$  extension  $\mathbb{F}_q(\mathbb{P}^1) \hookrightarrow \mathbb{F}_q(C)$ .

## Split Fibers and Obstructing Points

A point  $t \in \mathbb{P}^1(\mathbb{F}_q)$  corresponds to a line  $\ell_t$  through  $P$ . The fiber  $\pi_P^{-1}(t)$  is exactly the scheme-theoretic intersection  $C \cap \ell_t$ .

Therefore the following are equivalent:

- $\ell_t$  meets  $C$  in  $d$  distinct  $\mathbb{F}_q$ -points;
- $\pi_P^{-1}(t)$  consists of  $d$  distinct  $\mathbb{F}_q$ -points;
- the degree-1 place  $t$  of  $\mathbb{F}_q(\mathbb{P}^1)$  splits completely in the extension  $\mathbb{F}_q(C)/\mathbb{F}_q(\mathbb{P}^1)$ .

So an external point  $P$  is *obstructing* precisely when *no* degree-1 place of  $\mathbb{F}_q(\mathbb{P}^1)$  splits completely in the function field extension induced by  $\pi_P$ .

## Galois Closure, Geometric Monodromy, Arithmetic Monodromy

Let  $K = \mathbb{F}_q(\mathbb{P}^1)$  and  $L = \mathbb{F}_q(C)$ . Choose a Galois closure  $M/K$  of the finite separable extension  $L/K$ .

The *arithmetic monodromy group* of  $\pi_P$  is  $G_P^{\text{arith}} := \text{Gal}(M/K)$ .

Now extend scalars from  $\mathbb{F}_q$  to  $\overline{\mathbb{F}_q}$ . Equivalently, replace  $K$  by  $K\overline{\mathbb{F}_q} = \overline{\mathbb{F}_q}(\mathbb{P}^1)$ . The *geometric monodromy group* is  $G_P^{\text{geom}} := \text{Gal}(M\overline{\mathbb{F}_q}/\overline{\mathbb{F}_q}(\mathbb{P}^1))$ .

There is a natural exact sequence

$$1 \rightarrow G_P^{\text{geom}} \rightarrow G_P^{\text{arith}} \rightarrow \text{Gal}(\mathbb{F}_q/\mathbb{F}_q) \rightarrow 1$$

for a suitable finite constant field extension  $\mathbb{F}_{q^m}/\mathbb{F}_q$ .

Thus the quotient  $G_P^{\text{arith}}/G_P^{\text{geom}}$  measures possible *constant field enlargement* in the Galois closure. This enlargement is the main arithmetic obstruction to completely split  $\mathbb{F}_q$ -rational fibers.

## Effective Chebotarev over Function Fields

Assume the full constant field of the Galois closure  $M$  is exactly  $\mathbb{F}_q$ . The function field Chebotarev theorem implies that the number of completely split rational places is approximately

$$\frac{q}{|G_P^{\text{arith}}|}$$

up to an error term of size  $O(\sqrt{q})$ .

Hence, for fixed degree and sufficiently large  $q$ , complete splitting must occur unless the arithmetic monodromy is altered by constant field enlargement.

## Example: Quartics

For smooth quartics, the generic external behavior is expected to be

$$G_P^{\text{geom}} \cong S_4.$$

However, some exceptional points can have smaller geometric monodromy, notably

$$C_4, \quad A_4, \quad D_4.$$

Note that in the quartic case,  $G_P^{\text{geom}}$  and  $G_P^{\text{arith}}$  are subgroups of  $S_4$ . Since both  $S_4$  and  $D_4$  are self-normalizing in  $S_4$ , we know that no constant field enlargement occurs in these cases.

However, the  $A_4$  and  $C_4$  geometric monodromy groups can genuinely produce constant field enlargement.

These exceptional types are detected by special tangency configurations in the pencil through  $P$ .

The existence of a curve admitting external points satisfying these tangency conditions is quite rare. To prove such a statement regarding all quartic curves, we must work with the *moduli spaces* of curves.

## Genericity of Complete Arcs in Moduli

Let  $\mathcal{P}_d$  denote the moduli space of smooth plane curves of degree  $d$ . Its dimension is

$$\dim \mathcal{P}_d = \binom{d+2}{2} - 1 - \dim \text{PGL}_3 = \binom{d+2}{2} - 9 = \frac{d^2 + 3d - 16}{2}.$$

Now fix a monodromy type for projections from external points  $\pi_P$ . Let

$$\sigma_1, \dots, \sigma_r \in S_d$$

be the nontrivial local branch cycles recording tangent intersections. Define the index of a branch cycle by

$$\text{ind}(\sigma_i) = d - \#\{\text{cycles of } \sigma_i\}.$$

For a degree- $d$  projection of a smooth plane curve, Riemann–Hurwitz gives

$$\sum_{i=1}^r \text{ind}(\sigma_i) = d(d-1).$$

Assume now that the monodromy type under consideration forces  $\text{ind}(\sigma_i) \geq m \geq 2$  for every nontrivial local branch cycle. Then

$$r \leq \frac{d(d-1)}{2}.$$

The space of such projections with  $r$  branch points has dimension  $r - 3$  since the unordered branch divisor on  $\mathbb{P}^1$  depends on  $r$  parameters, and one quotient by  $\dim \text{PGL}_2 = 3$ .

Therefore every space of projection maps for this monodromy type has dimension at most

$$r - 3 \leq \frac{d(d-1)}{2} - 3.$$

Compare this with  $\dim \mathcal{P}_d$ . Their difference is

$$\frac{d^2 + 3d - 16}{2} - \left( \frac{d(d-1)}{2} - 3 \right) = 2d - 5 > 0 \quad (d \geq 3).$$

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